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A CLASS OF SIMPLE LIE GROUPS WHOSE QUASIREPRESENTATIONS WITH SMALL DEFECT CAN BE APPROXIMATED BY AN ORDINARY REPRESENTATION

A. I. Shtern

Abstract. We prove that every connected Lie group with finite center (as is well known, the semisimple groups of this kind admit no nontrivial real pseudocharacters) satisfies the following triviality theorem (which thus is not a characterization of amenable groups of this kind): every locally bounded finite-dimensional quasirepresentation of such a group with sufficiently small defect admits a close ordinary representation of the group.

§ 1. Introduction

For a preliminary information concerning quasirepresentations of simple Lie groups, see [1]. For the definitions needed below, see also [2].

Definition 1. The realization of a quasirepresentation of a group in a finite-dimensional space in the form given by formula (1) below and satisfying conditions (1)–(3) listed in the formulation of the main theorem below in §2 is referred to as a *canonical realization* of the quasirepresentation.

Let us recall some facts we need [2].

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Theorem A (immediate corollary to Theorem 3.3.14 in [2]). *Let G be a connected simply connected noncompact simple Lie group and let π be a quasirepresentation of G with sufficiently small defect in a finite-dimensional vector space E_π. Let*

$$\Theta(g)=\begin{pmatrix}\alpha(g) & 0 & 0 & \psi(g)\\ 0 & \beta(g) & 0 & 0\\ 0 & 0 & \Gamma(g) & 0\\ 0 & 0 & 0 & \delta(g)\end{pmatrix},\qquad g\in G,$$

be a canonical realization of π (according to part (4) of Theorem 3.3.14 in [2]*, such a canonical realization exists). The following assertions hold.*

(i) *The bounded representations α and δ are multiples of the identity representation of G.*

(ii) *If the center of the group G is finite, then the mapping Γ is a multiple of the identity representation of G.*

(iii) *If the center of the group G is finite, then $\psi = 0$.*

§ 2. Main Theorem: Formulation

Definition 2. Let G be a connected locally compact group, and let R be the radical of G. A character χ of R is said to be *central* if the relation $\chi(r)=\chi(g^{-1}rg)$ holds for all $g\in G$ and $r\in R$.

Theorem. *Let G be a connected Lie group with finite center and let π be a locally bounded finite-dimensional ε-quasirepresentation of G with sufficiently small ε. Let E_π^* be the space dual to E_π. Let L be the set of vectors $\xi\in E_\pi$ whose orbit $\{\pi(g)\xi \mid g\in G\}$ is bounded in E; let M be the set of functionals $f\in E_\pi^*$ whose orbit $\{\pi(g)^*f \mid g\in G\}$ is bounded in E_π^*; then L and the annihilator $M^\perp$ of M are π-invariant vector subspaces in E_π. Let us consider an increasing family of subspaces $\{0\}$, $L\cap M^\perp$, $M^\perp$, $L+M^\perp$, $E=E_\pi$ and write out the matrix $t(g)$ of the operator $\pi(g)$, $g\in G$, in the block form corresponding to the decomposition of the space E into the direct sum of subspaces $L\cap M^\perp$, $M^\perp\setminus(L\cap M^\perp)$, $L\setminus(L\cap M^\perp)$, and $E\setminus(L+M^\perp)$, where the symbol "$\setminus$" stands for the passage to a complementary subspace*:

$$(1)\qquad \Pi(g)=\begin{pmatrix}\alpha(g) & \varphi(g) & \sigma(g) & \tau(g)\\ 0 & \beta(g) & 0 & \rho(g)\\ 0 & 0 & \gamma(g) & \chi(g)\\ 0 & 0 & 0 & \delta(g)\end{pmatrix},\qquad g\in G.$$

($\Pi_{23}(g) = 0$, because L is invariant with respect to π.) Then the following assertions hold:

(1) *the mappings α, δ, γ, σ, and χ are bounded;*

(2) *the matrix-valued mappings Π_1 and Π_2 defined by the formulas $\Pi_1(g) = \begin{pmatrix} \alpha(g) & \varphi(g) \\ 0 & \beta(g) \end{pmatrix}$ and $\Pi_2(g) = \begin{pmatrix} \beta(g) & \rho(g) \\ 0 & \delta(g) \end{pmatrix}$ are representations of G;*

(3) *the mapping τ is a quasicocycle with respect to the representations t_1 and t_2, i.e., the mapping $(g,h) \mapsto \tau(gh) - \alpha(g)\tau(h) - \varphi(g)\rho(h) - \tau(g)\delta(h)$, $g,h \in G$, is bounded;*

(4) *a basis in the chain of subspaces $\{0\}$, $L \cap M^{\perp}$, $M^{\perp}$, $L + M^{\perp}$, $E = E_\pi$ can be chosen in such a way that the mappings σ and χ are small if the defect ε is;*

(5) *the mapping*

$$\Theta(g) = \begin{pmatrix} \alpha(g) & \varphi(g) & 0 & \Pi(g) \\ 0 & \beta(g) & 0 & \rho(g) \\ 0 & 0 & \Gamma(g) & 0 \\ 0 & 0 & 0 & \delta(g) \end{pmatrix}, \qquad g \in G,$$

approximating the quasirepresentation π, can be chosen to be an ordinary representation of G, and the representation Θ is continuous on the commutator subgroup of the radical R of G in the topology induced by that of G and continuous on the Levi subgroup of G/N in the intrinsic topology of the Lie group.

§ 3. Main Theorem: Proof

Proof of the theorem. Let G be a connected Lie group with finite center, let π be a locally bounded finite-dimensional ε-quasirepresentation of G with sufficiently small defect ε in a finite-dimensional normed linear space E, let S be a Levi subgroup of G, and let R be the radical of G.

Properties (1)–(3) are general specific features of construction (see [1]). Property (4) follows immediately from the very construction of the chain of subspaces $\{0\} \subseteq L \cap M^{\perp} \subseteq M^{\perp} \subseteq L + M^{\perp} \subseteq E = E_\pi$ (see the proof of Theorem 3.3.14 in [2]).

Property (5) and the existence of a realization of the mapping Θ described there follows from the fact that, according to the result cited above in Theorem A, the restriction of the canonical realization in question to the Levi subgroup S is an ordinary representation of S, and we may assume that the restriction of this realization to the radical R is an ordinary representation of S, since quasirepresentations of amenable (in particular, of solvable) groups with small defect admit close ordinary representations [1]. A simple observation shows that these representations of S and R are automatically combined into a representation of G.

Thus, every finite-dimensional quasirepresentation of a group in question with sufficiently small defect admits a close ordinary representation.

§ 4. Discussion

It seems that the result can be extended to almost connected locally compact groups.

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Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Institute of Systems Research (VNIISI),
Russian Academy of Sciences,
Moscow, 117312 Russia
E-mail: ashtern@member.ams.org, aishtern@mtu-net.ru